Multiple-mode wave solutions to display superpositions and collisions in nonlinear evolution equations

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We present a general method for obtaining multiple-mode waves (MMWs), which is introduced as a concept expressed in the form of nonlinear superpositions of single-mode waves (SMWs) with different wave speeds, for nonlinear evolution equations. The validity of the approach has been demonstrated using two wave equations. It is shown that MMWs may combine different types of SMWs such as periodic waves, kink waves, compactons, solitary waves, etc., to form more general solutions, which can be used to display the whole evolution process of interactions between different types of waves, especially to reveal the dynamic details of the wave patterns.

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INTRODUCTION

Directly seeking for exact solutions of nonlinear partial differential equations (NLPDEs) to describe many important phenomena in physics, biology, chemistry, etc., has become a central theme of perpetual interest in recent decades [1–5]. Many powerful methods have been presented for finding the solutions, such as the Backlund transformation, Darboux transformation, Painlevé method, tanh-coth method, homogeneous balance method, Jacobi elliptic function expansion method, and so on [6–10]. Furthermore, a great amount of activity has been concentrated on the various extensions and applications of the methods to simplify the routine of calculation. For example, a unified scheme called the mapping method has been developed to obtain Jacobi elliptic functions, solitons, and periodic functions for some NLPDEs [11,12]. The basic idea of the above-mentioned approaches is that, by introducing different types of *Ansätze*, the original partial differential equations can be transformed into a set of algebraic equations through balancing the same order of the *Ansatz*, which yields explicit expressions for the waves. However, only part of the special form of the waves can be derived by using most of these methods. In order to obtain all possible forms of the waves, recently bifurcation theory has been introduced to study the evolution of wave patterns with variation of the parameters [13–18].

Most of the above-mentioned methods as well as the waves obtained are based on the assumption that the solutions can be uniformly written in the form $u=u(\xi)$, with $\xi=x-ct$, which here we call single-mode waves (SMWs). These wave solutions cannot be used to describe the evolution details of the interactions between different waves. It is very difficult to give the details, in particular the analytical wave solutions, when two or even more waves meet together, since linear superposition theory is not suitable for nonlinear wave equations. In order to show the mechanism of the interactions, a collective variable (CV) approach has been presented to investigated the response of NLPDEs [19]. By employing an *Ansatz* which is an exact CV solution as well as the corresponding constraints, some special solutions for kink-kink and kink-antikink interactions or analogs for a few equations such as the sine-Gordon equation has been obtained [20]. Furthermore, based on a blend of transformations of independent variables and Hirota's method, the solutions can be expressed in terms of a Moloney and Hodnett type decomposition [21], which can be used to show the details of the different types of interaction between solitons. However, because of the additional constraints or special transformations taken, the above two approaches are restricted to describing the interactions between solitons. In order to shown the general mechanism of the interactions between different waves, here we introduce a wave solution, called the multiple-mode wave (MMW), which may simultaneously contain different wave speeds, i.e., $u=u(\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_1=x-c_1t$, $\xi_2=x-c_2t, \ldots, \xi_n=x-c_nt$. This type of solution may greatly help us to understand the mechanism of nonlinear superpositions of as well as the interactions between different forms of SMWs. In this paper, we will try to present a systematic method which can be easily employed using software such as MAPLE or MATHEMATICA to find the MMWs for nonlinear equations.

SKETCH OF THE METHOD

We assume that the MMW solutions for a given nonlinear equation

$$F(u,u_t,u_x,u_{xx}, \ldots)=0 \tag{1}$$

can be expressed in the form

$$u=\sum_{j_1+j_2+\cdots+j_n=0}^{k} p_{j_1j_2\cdots j_n}\phi_1^{j_1}\phi_2^{j_2}\cdots\phi_n^{j_n}, \tag{2}$$

with $\phi_i=\phi_i(\xi_i)=\phi_i(x+c_it)$ $(i=1,2,\ldots,n)$, representing n types of SMW evolving in the MMWs, which are the solutions of the following ordinary differential equations:

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$$\left(\frac{d\phi_i}{d\xi_i}\right)^2 = H_i(\phi_i) = \frac{1}{(\phi_i - h_i)^{m_i}}\sum_{j=0}^{k_i} a_{ij}\phi_i^j, \quad (3)$$

respectively, where the non-negative integers k, k_i, and m_i are assumed to be the related orders of the MMWs and SMWs, while $p_{j_1 j_2 \cdots j_n}$, h_i, c_i, and a_{ij} are constants to be determined later. Note that $d^2\phi_i/d\xi_i^2 = \frac{1}{2}dH_i(\phi_i)/d\phi_i$, based on which all high-order derivatives can be reduced to derivatives of the first order. Therefore, by substituting (2) and (3) into (1), we have

$$\frac{1}{f(\phi_1,\phi_2,\ldots,\phi_n)}\left[\sum_{i_1+i_2+\cdots+i_n=0}^{1}\ \sum_{j_1+j_2+\cdots+j_n=0,}^{M_2} q_{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} \times \phi_1^{j_1}\phi_2^{j_2}\cdots\phi_2^{j_n}\left(\frac{d\phi_i}{d\xi_i}\right)^{i_1}\left(\frac{d\phi_i}{d\xi_i}\right)^{i_2}\cdots\left(\frac{d\phi_i}{d\xi_i}\right)^{i_n}\right] = 0, \quad (4)$$

where $q_{i_1 i_2\cdots i_n j_1 j_2\cdots j_n}$ are functions of the physical parameters as well as the constants introduced above. Then the MMWs can be obtained by solving the algebraic equations $q_{i_1 i_2\cdots i_n j_1 j_2\cdots j_n}=0$, from which the relationships between the parameters and those constants can be determined.

To demonstrate the validity of the approach as well as to explore the typical structures of the multiple-mode waves, in the following, we consider two wave equations and focus on their two-mode wave solutions in the form $u=u(\phi_1,\phi_2)$, with $\phi_1=\phi_1(\xi_1)$, $\phi_2=\phi_2(\xi_2)$, $\xi_1=x-c_1t$, and $\xi_2=x-c_2t$, where c_1 and c_2 represent two different wave speeds for the SMWs.

MMWS OF A GENUINE SHALLOW WATER EQUATION

A lot of SMWs have been obtained based on various methods for the following $1+1$ nonlinear equation for unidirectional water waves with fluid velocity $u(x,t)$ as a genuine shallow water equation [22–24]:

$$m_t + c_0u_x + um_x + 2mu_x = -\gamma u_{xxx}, \quad (5)$$

where $m=u-\alpha^2u_{xx}$, which contains Korteweg–de Vries (KdV) solitons and Camassa-Holm (CH) peakons as limiting cases of the parameters [25]. With the help of MAPLE, for $k=2$, $k_i \leqslant 4$, and $m_i \leqslant 2$, we can find two cases for two-mode waves, which combine different SMWs.

Case A

The two-mode waves can be expressed in the form

$$u = p_{20}\phi_1^2 + p_{02}\phi_2^2 - 2h_1p_{20}\phi_1 - 2h_2p_{02}\phi_2 + 2\alpha^2(p_{20}a_2 + p_{02}b_2) - 2p_{20}h_1^2 - 2p_{02}h_2^2 - \frac{1}{2}c_0 - \frac{\gamma}{2\alpha^2}, \quad (6)$$

which is a superposition of the two SMWs ϕ_1 and ϕ_2, satisfying

$$\left(\frac{d\phi_1}{d\xi_1}\right)^2 = \frac{\phi_1^4 - 4h_1\phi_1^3 + 4\alpha^2a_2\phi_1^2 + (h_1^2-\alpha^2a_2)8h_1\phi_1 + 4\alpha^2a_0}{4\alpha^2(\phi_1-h_1)^2} \equiv \frac{F_{11}(\phi_1)}{4\alpha^2(\phi_1-h_1)^2}, \quad (7)$$

$$\left(\frac{d\phi_2}{d\xi_2}\right)^2 = \frac{\phi_2^4 - 4h_2\phi_2^3 + 4\alpha^2b_2\phi_2^2 + (h_2^2-\alpha^2b_2)8h_2\phi_2 + 4\alpha^2b_0}{4\alpha^2(\phi_2-h_2)^2} \equiv \frac{F_{12}(\phi_2)}{4\alpha^2(\phi_2-h_2)^2}, \quad (8)$$

respectively. Though the expressions in (7) and (8) to determine the two single-mode waves ϕ_1 and ϕ_2 are similar to each other, since all the constants, especially the wave speeds c_1 and c_2, can be taken arbitrarily, different forms of single-mode waves can be obtained associated with corresponding parameter conditions, which will be discussed in detail in the following. Therefore, it is not difficult to understand that, when different forms of the two single-mode waves ϕ_1 and ϕ_2 are taken, there exist different types of two-mode waves obtained through the same combination formula (6).

Because of the symmetry between $(d\phi_1/d\xi_1)^2$ and $(d\phi_1/d\xi_2)^2$, we need to discuss only all the possible SMWs for ϕ_1 as well as the existence conditions determined by Eq. (7), while the SMWs for ϕ_2 can be derived accordingly. The typical phase portraits for (7) are plotted in Fig. 1(a), where $y=d\phi_1/d\xi_1$. Each trajectory represents one possible solution for the system. Here we are interested only in bounded solutions. It is easy to check that only two types of bounded waves exist for (7), described below.

Kink compacton waves

For the conditions $\Delta_1>0$, $\Delta_2=0$, where $\Delta_1=3h_1^2-2\alpha^2a_2$, $\Delta_2=a_1^4a_2^2-2\alpha^2a_2h_1^2+h_1^4-\alpha a_0$, two real repeated zeros, denoted by $\phi_{10}^{(1)}$ and $\phi_{20}^{(1)}$ ($\phi_{10}^{(1)}=h_1+\sqrt{\Delta_1}$, $\phi_{20}^{(1)}=h_1-\sqrt{\Delta_1}$), can be obtained for $F_{11}(\phi_1)$, implying $F_{11}(\phi_1)=(\phi_1-\phi_{10}^{(1)})^2(\phi_1-\phi_{20}^{(1)})^2$. The kink compacton wave (KCW) associated with the trajectory L_1, denoted KCW_1, can be written in the form

$$\phi_1 = h_1, \quad \xi_1 - \xi_{10} \geqslant R_1,$$

$$\int_c^{\phi_1}\frac{(h_1-t)dt}{(a-t)(t-b)} = \frac{1}{2|\alpha|}(\xi_1-\xi_{10}), \quad \phi_1 \in [b,h_1), \quad \xi_1-\xi_{10} < R_1, \quad (9)$$

where $a=\phi_{10}^{(1)}$, $b=\phi_{20}^{(1)}$, ξ_{10} is the constant phase at $\phi_1=c$ ($b<c<a$), taken arbitrarily as the initial integral point, (9) can be further written in the form

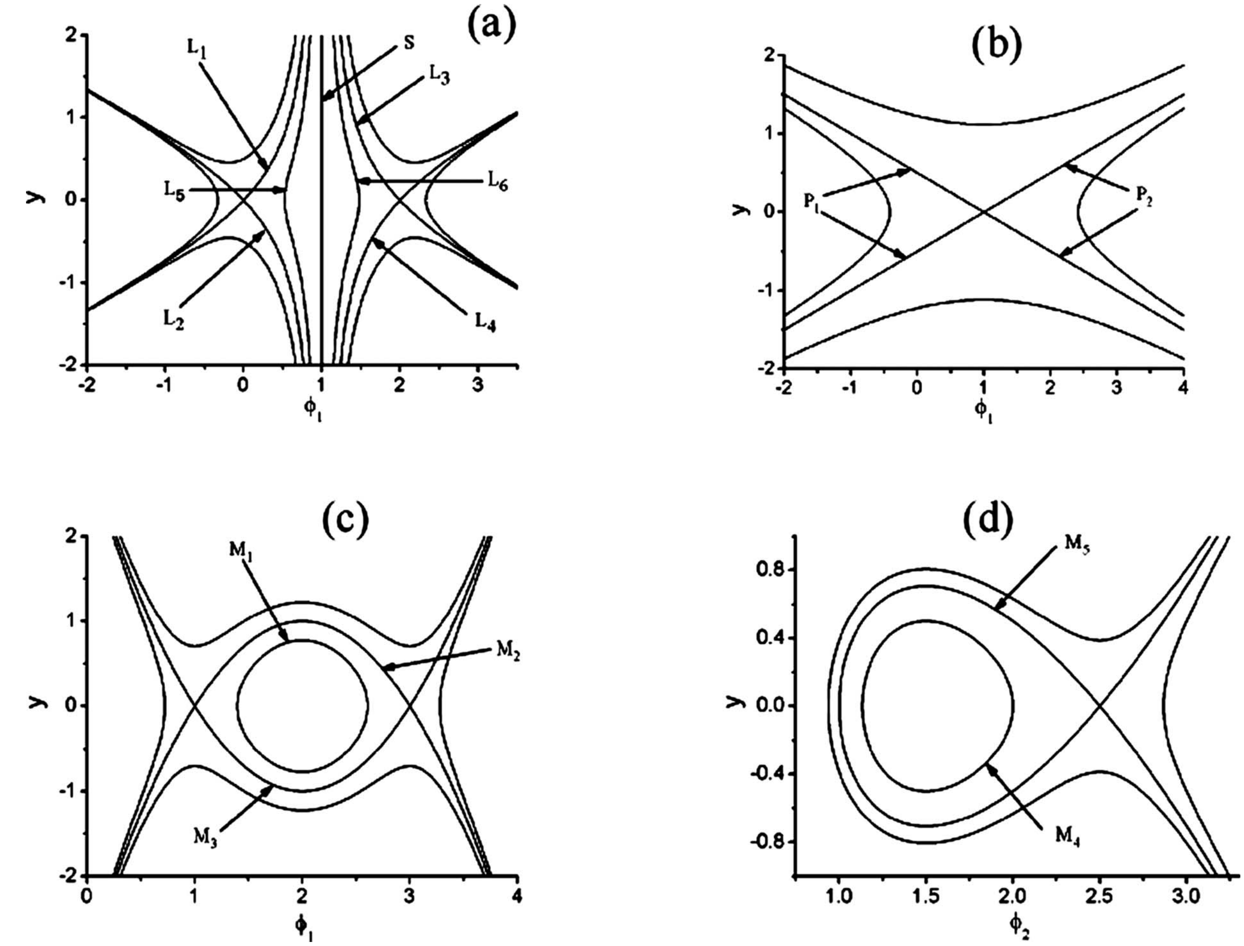

FIG. 1. Typical phase portraits: (a) for (7), (b) for (17), (c) for (21) and (d) for (22), where S denotes the singular line $\phi_1 = h_1$, the trajectories L_1, L_2, L_3 and L_4 correspond to kink-compactons, L_5 and L_6 correspond to compactons, P_1 and P_2 are related to peakons, M_1 and M_4 are associated with periodic waves, M_2 and M_3 correspond to kink waves, while M_5 is related to a solitary wave.

$$\phi_1 = h_1, \quad \xi_1 - \xi_{10} \geqslant R_1,$$

$$\frac{a-h_1}{a-b} \ln \frac{a-c}{a-\phi_1} + \frac{h_1-b}{a-b} \ln \frac{\phi_1-b}{c-b} = \frac{1}{2|\alpha|}(\xi_1 - \xi_{10}), \quad \phi_1 \in [b,h_1), \quad \xi_1 - \xi_{10} < R_1, \tag{10}$$

with

$$R_1 = 2|\alpha| \left| \frac{a-h_1}{a-b} \ln \frac{(a-c)}{(a-h_1)} + \frac{h_1-b}{a-b} \ln \frac{(h_1-b)}{(c-b)} \right|.$$

It can be checked that $\xi_1 \to -\infty$ for $\phi_1 \to b$, which means that for $\phi_1 \to b$ the wave displays the kink-wave structure, while for $\phi_1 \to h_1$, it behaves as a compacton. The explicit expressions for the kink compactons associated with the trajectories L_2, L_3, and L_4 can be obtained by a similar procedure, which we omit here for simplicity.

Compacton waves

For the conditions $\Delta_1 > 0$, $\Delta_2 \neq 0$, $\Delta_1 > |\Delta_2|$, four real zeros, expressed in the form

$$\phi_{10}^{(2)} = a = h_1 + \sqrt{\Delta_1 + \Delta_2}, \quad \phi_{20}^{(2)} = b = h_1 + \sqrt{\Delta_1 - \Delta_2},$$
$$\phi_{30}^{(2)} = c = h_1 - \sqrt{\Delta_1 - \Delta_2}, \quad \phi_{40}^{(2)} = d = h_1 - \sqrt{\Delta_1 + \Delta_2} \tag{11}$$

can be obtained for the function $F_{11}(\phi_1)$, which implies

$$F_{11}(\phi_1) = \frac{1}{4\alpha^2}(\phi_1 - a)(\phi_1 - b)(\phi_1 - c)(\phi_1 - d). \tag{12}$$

The compacton waves (CWs) associated with trajectories similar to L_5 (CW_1) can be expressed in the form

$$\phi_1 = h_1, \quad |\xi_1 - \xi_{10}| \geqslant R_2,$$

$$\int_c^{\phi_1} \frac{(h_1 - t)dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \pm \frac{1}{2|\alpha|}(\xi_1 - \xi_{10}), \quad \phi_1 \in [c,h_1], \quad |\xi_1 - \xi_{10}| < R_2, \tag{13}$$

where ξ_{10} is the constant phase at $\phi_1 = c$, which can be further integrated in the form

$$\phi_1 = h_1, \quad |\xi_1 - \xi_{10}| \geqslant R_2,$$

$$\left(h_1 - \frac{c\beta_1^2}{\beta^2}\right) gF(\varphi,k) - \frac{cg(\beta^2-\beta_2^2)}{\beta^2}\Pi(\varphi,\beta^2,k) = \pm\frac{1}{2|\alpha|}(\xi_1-\xi_{10}), \quad \phi_1 \in [c,h_1], \quad |\xi_1-\xi_{10}| < R_2, \tag{14}$$

where

$$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}, \quad g = \frac{2}{\sqrt{(a-c)(b-d)}},$$

$$\beta_1^2 = (b-c)/(b-d), \quad \beta_2^2 = [d(b-c)]/[c(b-d)],$$

$$\varphi = \sin^{-1}\sqrt{\frac{(b-d)(\phi_1-c)}{(b-c)(\phi_1-d)}},$$

$$\varphi_0 = \sin^{-1}\sqrt{\frac{(b-d)(h_1-c)}{(b-c)(h_1-d)}},$$

$$R_2 = \left| 2\alpha\left[\left(h_1-\frac{c\beta_1^2}{\beta^2}\right)gF(\varphi_0,k) - \frac{cg(\beta^2-\beta_2^2)}{\beta^2}\Pi(\varphi_0,\beta^2,k)\right]\right|. \tag{15}$$

Note that similar expressions for compactons associated with trajectories similar to L_6 can be obtained.

For this case, the two-mode wave solutions for Eq. (5) can be a nonlinear superposition according to (6) of two kink compactions, two compactons, or a kink compacton and a compacton, with different wave speeds.

Case B

The two-mode waves can be expressed in the form

$$u = p_{20}\phi_1^2 + p_{02}\phi_2^2 + 4\alpha^2 a_1 p_{20}\phi_1 - 2h_2 p_{02}\phi_2 + 2\alpha^2(p_{20}a_0 + p_{02}b_2) + 2\alpha^4 p_{20}a_1^2 - 2p_{02}h_2^2 - \frac{1}{2}c_0 - \frac{\gamma}{2\alpha^2}, \tag{16}$$

with the component waves ϕ_1 and ϕ_2 determined by

$$\left(\frac{d\phi_1}{d\xi_1}\right)^2 = a_0 + a_1\phi_1 + \frac{1}{4\alpha^2}\phi_1^2, \tag{17}$$

$$\left(\frac{d\phi_2}{d\xi_2}\right)^2 = \frac{\phi_2^4 - 4h_2\phi_2^3 + 4\alpha^2 b_2\phi_2^2 + (h_2^2-\alpha^2 b_2)8h_2\phi_2 + 4\alpha^2 b_0}{4\alpha^2(\phi_2-h_2)^2}, \tag{18}$$

respectively. The SMWs of ϕ_2 may be compactons or kink compactons, obtained using a procedure similar to the one above, while the bounded wave for ϕ_1 is defined as a peakon wave (PW), written in the form $\phi_1 = D\exp[-|(x-c_1t)/2\alpha|] - 2\alpha^2 a_1$ for $\alpha^2 a_1^2 - a_0 = 0$ and, particular, $\phi_1 = D\exp[-|(x-c_1t)/2\alpha|]$ for $a_0=0$, $a_1=0$, corresponding to the two typical trajectories P_1 and P_2 in Fig. 1(b). Therefore, two-mode waves can be a nonlinear superposition between a peakon and a compacton or a kink compacton.

MMWS OF VAKHNENKO EQUATION

Similarly, many types of MMWs can be derived for the Vakhnenko equation [26], written as

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0, \tag{19}$$

which describes gravity waves propagating down a channel under the influence of the Coriolis force [27]. Here we give only one typical case for the two-mode wave solution:

$$u = p_{20}\phi_1^2 + p_{10}\phi_1 + p_{01}\phi_2 + p_{00}, \tag{20}$$

with two SMWs ϕ_1 and ϕ_2 defined by

$$\left(\frac{d\phi_1}{d\xi_1}\right)^2 = -\frac{p_{20}}{6}F_{21}(\phi_1) = -\frac{p_{20}}{6}\left(\phi_1^4 + \frac{2p_{10}}{p_{20}}\phi_1^3 + \frac{6p_{20}b_2 + 12p_{00}p_{20} + 3p_{10}^2}{4p_{20}^2}\phi_1^2 + \frac{p_{10}(6p_{20}b_2 - p_{10}^2 + 12p_{00}p_{20})}{4p_{20}^3}\phi_1 - \frac{6a_0}{p_{20}}\right), \tag{21}$$

$$\left(\frac{d\phi_2}{d\xi_2}\right)^2 = -\frac{2p_{01}}{3}F_{22}(\phi_2) = -\frac{2p_{01}}{3}\left(\phi_2^3 - \frac{3b_2}{2p_{01}}\phi_2^2 - \frac{48p_{00}p_{20}^2(b_2+p_{00}) - 6p_{20}p_{10}^2(b_2+2p_{00}) + 96p_{20}^3 a_0 + p_{10}^4}{16p_{01}^2 p_{20}^2}\phi_2 - \frac{3b_0}{2p_{01}}\right), \tag{22}$$

respectively. We first consider SMWs of ϕ_1 defined in (21), the typical phase portrait of which is plotted in Fig. 1(c). Three real zeros, denoted by $\phi_1^{(1)}$, $\phi_1^{(2)}$, and $\phi_1^{(3)}$, can be obtained for the function $dF_{21}(\phi_1)/d\phi_1$ for $\delta_1(\delta_2+\delta_3) \geqslant 0$, written as

$$\phi_1^{(1)} = -\frac{p_{10}}{2p_{20}},$$

$$\phi_1^{(2,3)} = \frac{2p_{10}p_{02}p_{20}(-p_{01}^2 + 8p_{02}p_{00}) \pm 2p_{10}^3p_{02}^2 + \sqrt{\delta_1(\delta_2+\delta_3)}}{4p_{02}p_{20}(p_{10}^2p_{02} + p_{01}^2p_{20} - 8p_{02}p_{20}p_{00})}, \quad (23)$$

where

$$\delta_1 = 2p_{02}(-p_{10}^2p_{02} - p_{01}^2p_{20} + 8p_{02}p_{20}p_{00}),$$

$$\delta_2 = -48p_{02}^2p_{20}^2(2p_{02}b_0 - 2p_{20}b_0 + p_{00}^2) - 2p_{10}^2p_{02}^2(-12p_{20}p_{00} + p_{10}^2),$$

$$\delta_3 = p_{01}^2p_{20}(12p_{02}p_{20}p_{00} - 3p_{10}^2p_{02} - p_{01}^2p_{20}), \quad (24)$$

at which the function $F_{21}(\phi_1)$ reaches the extreme values, expressed by $g_1^{(1)}=F_{21}(\phi_1^{(1)})$, $g_1^{(2)}=F_{21}(\phi_1^{(2)})=g_1^{(3)}=F_{21}(\phi_1^{(3)})$. Here we consider only the SMWs for $p_{20}<0$, while the situations for $p_{20}>0$ can be discussed using a similar procedure. Different forms of single-mode waves can be observed with the corresponding existence conditions; the details are described below.

Periodic waves

When $g_1^{(1)}>0$, $g_1^{(2)}<0$, there exist four real zeros for the function $F_{21}(\phi_1)$, denoted by a, b, c, and d, $(a>b>c>d)$, which implies $F_{21}(\phi_1)=(\phi_1-a)(\phi_1-b)(\phi_1-c)(\phi_1-d)$. Only periodic wave solutions associated with the trajectories similar to M_1 in Fig. 1(c) can be obtained, written in the form

$$\int_c^{\phi_1} \frac{dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \pm\sqrt{-\frac{p_{20}}{6}}(\xi-\xi_{10}) + nT_1, \quad \phi_1 \in [c,b], \quad (25)$$

where ξ_{10} denotes the constant phase of the waves at $\phi_1=c$, and T_1 represents the period of the waves, which can be written in the integrated form

$$\frac{2}{\sqrt{(a-c)(b-d)}}F(\varphi,k) = \pm\sqrt{-\frac{p_{20}}{6}}(\xi-\xi_{10}) + nT_1, \quad (26)$$

with

$$\varphi = \sin^{-1}\sqrt{\frac{(b-d)(\phi_1-c)}{(b-c)(\phi_1-d)}}, \quad k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)},$$

$$T_1 = \sqrt{-\frac{6}{p_{20}}}\frac{4}{\sqrt{(a-c)(b-d)}}F\left(\frac{\pi}{2},k\right). \quad (27)$$

Kink waves

In particular, for $g_1^{(1)}>0$, $g_1^{(2)}=0$, two repeated real zeros, denoted by a, and b $(a>b)$, can be observed for the function $F_{21}(\phi_1)$, which means $F_{21}(\phi_1)=(\phi_1-a)^2(\phi_1-b)^2$. Two types of kink waves (KWs) exist, corresponding to the trajectories M_2 (KW_1) and M_3 (KW_2), expressed by

$$\frac{1}{a-b}\left(\ln\left|\frac{\phi_{10}-a}{\phi_{10}-b}\right| - \ln\left|\frac{\phi_1-a}{\phi_1-b}\right|\right) = \sqrt{-\frac{p_{20}}{6}}(\xi-\xi_{10}), \quad (28)$$

$$\frac{1}{a-b}\left(\ln\left|\frac{\phi_{10}-a}{\phi_{10}-b}\right| - \ln\left|\frac{\phi_1-a}{\phi_1-b}\right|\right) = -\sqrt{-\frac{p_{20}}{6}}(\xi-\xi_{10}), \quad (29)$$

respectively, where $\phi_1 \in [b,a]$. ξ_{10} is the constant phase at which $\phi_1=\phi_{10}\in(b,a)$, taken arbitrarily as the initial integration point. From the integration result in (28), it can be seen that, as $\phi_1\to a$, ξ_1 tends to positive infinity logarithmically, while for $\phi_1\to b$, ξ_1 tends to negative infinity logarithmically, and similarly for (29), which demonstrates that both the solutions in (28) and (29) are kink waves.

We now turn to the SMWs of (22); the typical phase portrait is plotted in Fig. 1(d). Obviously, there exist two zeros for the function $dF_{22}(\phi_2)/d\phi_2$, written as

$$\phi_2^{(1)} = \frac{1}{4p_{01}p_{20}}\left(2b_2p_{20} + \frac{1}{3}\sqrt{\delta}\right),$$

$$\phi_2^{(2)} = \frac{1}{4p_{01}p_{20}}\left(2b_2p_{20} - \frac{1}{3}\sqrt{\delta}\right), \quad (30)$$

where $\delta=288a_0p_{20}^3+36(4b_2p_{00}+4p_{00}^2+b_2^2)p_{20}^2-18(2p_{00}+b_2)p_{10}^2p_{20}+3p_{10}^4$, at which the function $F_{22}(\phi_2)$ reaches its extreme values, denoted by $g_2^{(1)}=F_{22}(\phi_2^{(1)})$ and $g_2^{(2)}=F_{22}(\phi_2^{(2)})$. It can be checked that no bounded wave can be obtained for $\delta<0$. Here we consider only the case for the conditions $p_{01}<0$, $p_{20}<0$, $\delta>0$. Noting that $g_2^{(1)}-g_2^{(2)}=-\sqrt{\delta}\delta/(144p_{01}^3p_{20}^3)$, and we find $\phi_2^{(1)}>\phi_2^{(2)}$, $g_2^{(1)}<g_2^{(2)}$.

Periodic wave

When $g_2^{(1)}<0$, $g_2^{(2)}>0$, three real zero points for the function $F_{22}(\phi_2)$ can be observed, which implies $F_{22}(\phi_2)=(\phi_2-a)(\phi_2-b)(\phi_2-c)$. Only periodic waves associated with trajectories similar to M_4 in Fig. 1(d) can be obtained, written in the form

$$\int_{\phi_2}^{b}\frac{dt}{\sqrt{(a-t)(b-t)(t-c)}} = \pm\sqrt{-\frac{2p_{01}}{3}}(\xi-\xi_{20}) + nT_2, \quad (31)$$

with the integration

$$\frac{2}{\sqrt{a-c}}F(\varphi,k) = \pm\sqrt{-\frac{2p_{01}}{3}}(\xi-\xi_{20}) + nT_2, \quad (32)$$

where ξ_{20} is the constant phase at which $\phi_2=b$, T_2 represents the period of the waves, and

$$\varphi = \sin^{-1}\sqrt{\frac{(a-c)(b-\phi_2)}{(b-c)(a-\phi_2)}}, \quad k^2 = \frac{(b-c)}{(a-c)},$$

$$T_2 = \frac{4}{\sqrt{a-c}}\sqrt{-\frac{3}{2p_{01}}}F\left(\frac{\pi}{2},k\right). \tag{33}$$

Solitary waves

When $g_2^{(1)}=0$, $g_2^{(2)}>0$, a simple real zero as well as a repeated real zero can be found for $F_{22}(\phi_2)$, implying $F_{22}(\phi_2)=(\phi_2-a)^2(\phi_2-c)$. Only a solitary wave as a bounded solution corresponding to the trajectory M_5 can be obtained, expressed by

$$\frac{1}{\sqrt{a-c}}\ln\left(\frac{(\sqrt{a-c}+\sqrt{\phi_2-c})(\sqrt{a-c}-c)}{(\sqrt{a-c}-\sqrt{\phi_2-c})(\sqrt{a-c}+c)}\right) = \pm\sqrt{-\frac{2p_{01}}{3}}(\xi-\xi_{20}), \tag{34}$$

where ξ_{20} is the constant phase at which $\phi_2=c$. It is easy to check that $\xi\to\mp\infty$ when $\phi_2\to a$, which means the waves in (34) are solitons.

For this case, the two-mode waves can be nonlinear superpositions according to (20) between two periodic waves, a periodic wave and a kink wave, a periodic wave and a kink wave, or a solitary wave and a kink wave.

DISCUSSION

The great importance of the multiple-mode waves lies in the fact that this type of solution can clearly display the whole evolution process of the interactions between different types of waves. For example, when several waves coexist, the MMWs can describe the details before or after the waves meet, especially the shapes of superpositions. Different wave patterns can be obtained by employing numerical simulation of MMWs changing with temporal as well as spatial variables. This demonstrates that the solitons may keep their structures before and after interactions with other types of wave solutions.

We would like to mention two facts related to the two-mode wave forms. First, by taking certain values of the constants in the superposition formulas eliminating ϕ_1 or ϕ_2 from the solution, SMWs can be obtained for the equations. Second, only the lower order of the two-mode waves as well as lower order superpositions have been presented as examples to demonstrate the validity of the approach; more forms of two-mode wave solutions can be obtained when higher-order two-mode waves as well as higher-order superpositions are taken. Furthermore, in the computation of the MMWs, we find that the higher-order two-mode waves contain the lower-order two-mode waves as special cases.

Obviously, though only two-mode wave solutions have been considered as examples for MMWs for the two equations, more complicated multiple-mode waves which combine three or even more single-mode waves can be derived by employing similar procedures as described in this work. The method can also be used for finding MMWs in high-dimensional systems to investigate the superpositions of as well as the interactions between SMWs, which may help us to understand the complicated dynamics of nonlinear evolution equations.

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[1] Sadhan K. Adhikari, Phys. Lett. A **346**, 179 (2005).
[2] E. Meletlidou and P. G. L. Leach, Chaos, Solitons Fractals **34**, 903 (2007).
[3] Jaehoon Lee, Opt. Fiber Technol. **11**, 46 (2005).
[4] J. Krishnan, O. Runborg, and I. G. Kevrekidis, Comput. Chem. Eng. **28**, 557 (2004).
[5] A. H. A. Ali, A. A. Soliman, and K. R. Raslan, Phys. Lett. A **368**, 299 (2007).
[6] T. Alagesan, Y. Chung, and K. Nakkeeran, Chaos, Solitons Fractals **21**, 63 (2004).
[7] S. Roy Choudhury, Chaos, Solitons Fractals **27**, 139 (2006).
[8] Abdul-Majid Wazwaz, Appl. Math. Comput. **188**, 1930 (2007).
[9] S. A. El-Wakil, E. M. Abulwafa, A. Elhanbaly, and M. A. Abdou, Chaos, Solitons Fractals **33**, 1512 (2007).
[10] Zhang Huiqun, Commun. Nonlinear Sci. Numer. Simul. **12**, 627 (2007).
[11] M. A. Abdou and Sheng Zhang, Commun. Nonlinear Sci. Numer. Simul. (to be published).
[12] Ranchao Wu and Jianhua Sun, Chaos, Solitons Fractals **31**, 70 (2007).
[13] L. Zhang and J. Li, Chaos, Solitons Fractals **17**, 941 (2003).
[14] Zhengdi Zhang and Qinsheng Bi, Chaos, Solitons Fractals **24**, 631 (2005).
[15] Qinsheng Bi, Phys. Lett. A **360**, 574 (2007).
[16] Stefan C. Mancas and S. Roy Choudhury, Math. Comput. Simul. **74**, 266 (2007).
[17] Qinsheng Bi, Phys. Lett. A **352**, 227 (2006).
[18] Qinsheng Bi, Chaos, Solitons Fractals **31**, 417 (2007).
[19] C. D. Ferguson and C. R. Willis, Physica D **199**, 283 (1998).
[20] K. Forinash and C. R. Willis, Physica D **149**, 95 (2001).
[21] A. J. Morrison and E. J. Parkes, Chaos, Solitons Fractals **16**, 13 (2003).
[22] R. Camassa and D. D. Holm, Phys. Rev. Lett. **71**, 1661 (1993).
[23] M. Alber, R. Camassa, D. D. Holm, and J. E. Marsden, Lett. Math. Phys. **32**, 137 (1994).
[24] H. R. Dullin, G. A. Gottwald, and D. D. Holm, Physica D **190**, 1 (2004).
[25] Qinsheng Bi, Phys. Lett. A **344**, 361 (2005).
[26] E. Yusufoglu and A. Bekir, Chaos, Solitons Fractals (to be published).
[27] V. O. Vakhnenko and E. J. Parkes, Nonlinearity **11**, 1457 (1998).